

ON THE INVERSE LIMIT OF EUCLIDEAN N-SPHERES⁽¹⁾

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In [1] Bing constructed a 1-dimensional hereditarily indecomposable continuum which is the inverse limit of a sequence of circles C_i and such that the maps $f_i: C_i \rightarrow C_{i-1}$ were of degree 1. In this paper we show that the dimension cannot be raised, i.e., if $S = \text{Lim}(S_i^N, f_i)$ where S_i^N is an N -sphere and f_i is essential, then S is *not* hereditarily indecomposable. In doing so we further generalize Bing's definition of " ϵ -crooked." Lemma 2 shows how to construct hereditarily indecomposable continua by making the bonding maps sufficiently crooked. In Lemma 3 we get a necessary crookedness condition on the bonding maps if the limit space is to be hereditarily indecomposable. The remainder of the paper is devoted to proving that in the case of N -spheres ($N > 1$) this condition cannot be satisfied.

DEFINITIONS AND NOTATION. Let X_i be a sequence of compact metric spaces, and for $i \geq 2$ let f_i be a map of X_i into X_{i-1} . Then the subspace⁽²⁾⁽³⁾

$$S = \left\{ z \in \prod_1^\infty X_i \mid f_{ij}(z_j) = z_i \right\}$$

of $\prod_1^\infty X_i$ is the *inverse limit space* of the *inverse system* (X_i, f_i) .

If d_i is the diameter of X_i then a metric for $\prod_1^\infty X_i$ (and hence for S) is given by $|z - z'| = \sum_1^\infty 2^{-i} d_i^{-1} |z_i - z'_i|$.

Let $f: X \rightarrow Y$ where X, Y are metric spaces. Then for $\epsilon > 0$ ⁽⁴⁾,

$$L(\epsilon, f) = \sup \left\{ \delta \mid \begin{array}{l} x, y \in X \text{ and } |x - y| < \delta \\ \text{implies } |f(x) - f(y)| < \epsilon \end{array} \right\}.$$

If X is compact then $L(\epsilon, f) > 0$ for all $\epsilon > 0$.

Let $f: X \rightarrow Y$ where X is a topological space and Y is a metric space. Let $\epsilon > 0$. Then f is ϵ -crooked if for each path $g: I \rightarrow X$ ⁽⁵⁾ there exist real numbers t_1, t_2 such that $0 \leq t_1 \leq t_2 \leq 1$ and:

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⁽²⁾ $f_{ij} = f_{i+1}f_{i+2} \cdots f_i, f_{ii} = 1$.

⁽³⁾ If $z \in \prod_1^\infty X_i$, then z_i denotes the i th coordinate of z . Hence $z = (z_i)$.

⁽⁴⁾ $|x - y|$ denotes the distance from x to y .

⁽⁵⁾ I denotes the unit interval $[0, 1]$.

$$\begin{aligned} |fg(0) - fg(t_2)| &< \epsilon, \\ |fg(t_1) - fg(1)| &< \epsilon. \end{aligned}$$

A metric space Y is ϵ -crooked if the identity map $1: Y \rightarrow Y$ is ϵ -crooked⁽⁶⁾.

A continuum M is *indecomposable* if it is not the union of two proper subcontinua. If every subcontinuum of M is indecomposable then M is *hereditarily indecomposable*.

LEMMA 1. *Let K be a metric space. Suppose $K_1, K_2, \dots$ is a sequence of Peanian continua in K such that $K_1 \supset K_2 \supset \dots$. Then $\bigcap_1^\infty K_n$ is hereditarily indecomposable if and only if there is a null sequence (ϵ_n) of positive real numbers such that K_n is ϵ_n -crooked.*

Proof of sufficiency. Suppose M is a subcontinuum of $\bigcap_1^\infty K_n$ and $M = A \cup B$, where A, B are proper subcontinua of M . Let $a \in A - B$, $b \in B - A$. Then there is an n such that $\epsilon_n < \min(|a - B|, |b - A|)$ ⁽⁷⁾. Let O_A, O_B be connected open subsets of K_n such that $A \subset O_A$, $B \subset O_B$, $|a - \bar{O}_B| > \epsilon_n$, and $|b - \bar{O}_A| > \epsilon_n$. Let $x \in O_A \cap O_B$. Then there are paths $\alpha, \beta: I \rightarrow K_n$ such that $\alpha(0) = a$, $\alpha(1) = x$, $\beta(0) = x$, $\beta(1) = b$, $\alpha(I) \subset O_A$, and $\beta(I) \subset O_B$. Let $g: I \rightarrow K$ where

$$g(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq 1/2, \\ \beta(2t - 1), & 1/2 \leq t \leq 1. \end{cases}$$

Since K_n is ϵ_n -crooked there exist t_1, t_2 such that $t_1 \leq t_2$, $|a - g(t_2)| < \epsilon_n$ and $|g(t_1) - b| < \epsilon_n$. Hence $t_2 < 1/2$ and $t_1 > 1/2$. But this contradicts the fact that $t_1 \leq t_2$.

Proof of necessity. Suppose $\bigcap_1^\infty K_n$ is hereditarily indecomposable and for some $\epsilon > 0$ there is a sequence n_i such that K_{n_i} is not ϵ -crooked. Then for each i there is a path $\alpha_i: I \rightarrow K_{n_i}$, and real numbers t_i such that $|\alpha_i([0t_i]) - \alpha_i(1)| \geq \epsilon$ and $|\alpha_i([t_i, 1]) - \alpha_i(0)| \geq \epsilon$ ⁽⁸⁾. Let $A_{n_i} = \alpha_i([0t_i])$, $B_{n_i} = \alpha_i([t_i, 1])$, $M_{n_i} = \alpha_i(I)$. Then $\limsup A_{n_i}$, $\limsup B_{n_i}$ are proper subcontinua of the continuum $\limsup M_{n_i}$, and $\limsup M_{n_i} = \limsup A_{n_i} \cup \limsup B_{n_i}$.

LEMMA 2. *Let $S = \lim(X_i, f_i)$ where the X_i are Peanian continua with diameters d_i . Suppose for all n , f_n is ϵ_n -crooked where $\epsilon_n < \min_{i < n-1} L(2^{-n}d_i, f_{i-n-1})$. Then S is hereditarily indecomposable.*

Proof. Let $K_n = \{z \in \prod_1^\infty X_i \mid f_{ij}(z_j) = z_i \text{ for } i < j \leq n\}$. Then K_n is homeomorphic to X_n and $S = \bigcap_{n=1}^\infty K_n$. Now K_n is 2^{3-n} -crooked. For if $g: I \rightarrow K_n$ is a path in K_n , then⁽⁹⁾

⁽⁶⁾ It may be noted that both definitions of ϵ -crooked (for maps and for spaces) depend on the given metric.

⁽⁷⁾ $|x - Y|$ denotes the distance from x to Y .

⁽⁸⁾ We may assume without loss of generality that $\alpha_i(t_i)$ converges.

⁽⁹⁾ π_n denotes the map collapsing each point of $\prod_1^\infty X_i$ onto its n th coordinate.

$$I \xrightarrow{g} K_n \xrightarrow{\pi_n} X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{i,n-1}} X_i.$$

Since f_n is ϵ_n -crooked, there exist real numbers t_1, t_2 such that $0 \leq t_1 \leq t_2 \leq 1$, $|f_n \pi_n g(0) - f_n \pi_n g(t_2)| < \epsilon_n$, and $|f_n \pi_n g(t_1) - f_n \pi_n g(1)| < \epsilon_n$. Let $\alpha = g(0), \beta = g(t_2) \cdot (\alpha, \beta \in K_n)$. Then $|f_n \pi_n(\alpha) - f_n \pi_n(\beta)| < \epsilon_n$, i.e. $|\alpha_{n-1} - \beta_{n-1}| < \epsilon_n$. Since $\epsilon_n < L(2^{-n}d_i, f_{i,n-1})$, $|\alpha_i - \beta_i| < 2^{-n}d_i$ for $i = 1, 2, \dots, (n-2)$. Hence

$$\begin{aligned} |\alpha - \beta| &= \sum_1^{\infty} 2^{-i} d_i^{-1} |\alpha_i - \beta_i| \leq \sum_1^{n-2} 2^{-i} d_i^{-1} |\alpha_i - \beta_i| + 2^{2-n} \\ &\leq \sum_1^{n-2} 2^{-i} d_i^{-1} 2^{-n} d_i + 2^{2-n} \\ &\leq 2^{-n} \sum_1^{n-2} 2^{-i} + 2^{2-n} \\ &< 2^{-n} + 2^{2-n} \\ &< 2^{3-n}. \end{aligned}$$

Hence $|g(0) - g(t_2)| < 2^{3-n}$. Similarly, $|g(t_1) - g(1)| < 2^{3-n}$. Hence K_n is 2^{3-n} -crooked. But $S = \bigcap_{n=1}^{\infty} K_n$, so by Lemma 1, S is hereditarily indecomposable.

LEMMA 3. *Let $S = \lim(X_i, f_i)$ where the X_i are Peanian continua and S is hereditarily indecomposable. Then for any $\epsilon > 0$ there is an n such that f_{1n} is ϵ -crooked.*

Proof. By Lemma 1 there is an n such that K_n is $2^{-1}\epsilon d_1^{-1}$ -crooked. Let $g: I \rightarrow X_n$ be a path in X_n . Let $x_i \in X_i$ for $i = n+1, n+2, \dots$. Let $\bar{g}: I \rightarrow K_n$ by

$$\bar{g}(t) = (f_{1n}g(t), f_{2n}g(t), \dots, g(t), x_{n+1}, x_{n+2}, \dots).$$

Since K_n is $2^{-1}\epsilon d_1^{-1}$ -crooked there exist $t_1 \leq t_2$ such that $|\bar{g}(0) - \bar{g}(t_2)| < 2^{-1}d_1^{-1}\epsilon$, $|\bar{g}(t_1) - \bar{g}(1)| < 2^{-1}d_1^{-1}\epsilon$. Now $|\bar{g}(0) - \bar{g}(t_2)| = \sum_{i=1}^n 2^{-i} d_i^{-1} |f_{in}g(0) - f_{in}g(t_2)|$. Hence $\sum_{i=1}^n 2^{-i} d_i^{-1} |f_{in}g(0) - f_{in}g(t_2)| < 2^{-1}d_1^{-1}\epsilon$. In particular

$$2^{-1}d_1^{-1} |f_{1n}g(0) - f_{1n}g(t_2)| < 2^{-1}d_1^{-1}\epsilon, \text{ or } |f_{1n}g(0) - f_{1n}g(t_2)| < \epsilon.$$

Similarly $|f_{in}g(t_1) - f_{in}g(1)| < \epsilon$. Hence f_{1n} is ϵ -crooked.

LEMMA 4. *Let Y be a metric space, X a topological space, and f, g two maps of X into Y such that $\|f - g\| < \delta$. Then if f is ϵ -crooked, g is $(\epsilon + 2\delta)$ -crooked.*

Proof. Suppose $\alpha: I \rightarrow X$ is a path in X . Since f is ϵ -crooked there exist $t_1 \leq t_2$ such that $|f\alpha(0) - f\alpha(t_2)| < \epsilon$, $|f\alpha(t_1) - f\alpha(1)| < \epsilon$. Hence

$$\begin{aligned} |g\alpha(0) - g\alpha(t_2)| &\leq |g\alpha(0) - f\alpha(0)| + |f\alpha(0) - f\alpha(t_2)| + |f\alpha(t_2) - g\alpha(t_2)| \\ &< \|g - f\| + \epsilon + \|g - f\| \\ &< 2\delta + \epsilon. \end{aligned}$$

Similarly $|\alpha(t_1) - \alpha(1)| < \epsilon + 2\delta$.

Let T be a triangulation of the N -sphere S^N ($N > 1$) and suppose $\bar{f}: T^n \rightarrow T^m$ is a chain map of the n th barycentric subdivision of T onto the m th barycentric subdivision of T . Let $f: |T^n| \rightarrow |T^m|$ be the induced map of S^N upon itself and suppose f is essential. Finally let σ be a N -simplex of T . An open subset o of S^N is said to be *inessential* mod $|\sigma|$ if there is a homotopy $F: S^N \times I \rightarrow S^N$ such that

- (a) $F_0 = f$,
- (b) $F_t|_{(S^N - o)} = f$,
- (c) $F_1(o) \subset |\sigma|$.

Otherwise o is *essential* mod $|\sigma|$.

LEMMA 5. *Let o be essential mod $|\sigma|$. Then if F is any homotopy of $S^N \times I$ into S^N satisfying the above conditions (a) and (b), $F_1(o) \supset S^N - |\sigma|$.*

Proof. Suppose $x_0 \in S^N - [|\sigma| \cup F_1(o)]$. Since $|\sigma|$ is a strong deformation retract of $S^N - x_0$, there is a homotopy $G: (S^N - x_0) \times I \rightarrow S^N$ such that $G_0 = 1$, $G_t|_{|\sigma|} = 1$, and $G_1(S^N - x_0) \subset |\sigma|$. Let $H: S^N \times I \rightarrow S^N$ by:

$$H(x, t) = \begin{cases} F(x, 2t), & 0 \leq t \leq 1/2, \\ f(x), & 1/2 \leq t \leq 1, x \notin o, \\ G(F_1(x), 2t - 1), & 1/2 \leq t \leq 1, x \in o. \end{cases}$$

Then $H_0 = f$, $H_t|_{(S^N - o)} = f$, and $H_1(o) = G_1(F_1(o)) \subset G_1(S^N) \subset |\sigma|$. But this contradicts the assumption that o is essential mod $|\sigma|$.

LEMMA 6. *Suppose o_1, o_2 are disjoint open subsets of S^N . Then if each is inessential mod $|\sigma|$ so is their union.*

Proof. Let F, G be the homotopies corresponding to o_1, o_2 respectively. Let $H: S^N \times I \rightarrow S^N$ by:

$$H(x, t) = \begin{cases} F(x, 2t), & 0 \leq t \leq 1/2, \\ G(x, 2t - 1), & 1/2 \leq t \leq 1, x \in o_1, \\ F(x, 1), & 1/2 \leq t \leq 1, x \in o_2. \end{cases}$$

LEMMA 7. *If X is a component of $f^{-1}(|\sigma|)$, then at least one complementary domain of X is essential mod $|\sigma|$.*

Proof. Let $o_1, o_2, \dots, o_r$ be the complementary domains of X . If each o_i is inessential then by Lemma 6 so is their union $S^N - X$. Hence there is a homotopy $F: S^N \times I \rightarrow S^N$ such that $F_0 = f$, $F_t|_X = f$, and $F_1(S^N - X) \subset |\sigma|$. But then $F_1(S^N) \subset |\sigma|$. Hence F_1 is inessential. But F_1 is homotopic to f and f is essential.

LEMMA 8. *Suppose $\sigma_1, \sigma_2, \sigma_3$ are three N -simplexes of T whose geometric realizations are pairwise disjoint. Then there is an integer i_0 ($i_0 = 1, 2, 3$), a component X of $f^{-1}(|\sigma_{i_0}|)$, and a complementary domain o of X such that:*

(a) o is essential mod $|\sigma_{i_0}|$.

(b) If Y is any component of $f^{-1}(|\sigma_i|)$ ($i=1, 2, 3$) in o and o' is a complementary domain of Y lying in o , then o' is inessential mod $|\sigma_i|$.

Proof. Let X_1 be a component of $f^{-1}(|\sigma_1|)$. By Lemma 7, X_1 has a complementary domain o_1 which is essential mod $|\sigma_1|$. If the pair (X_1, o_1) does not satisfy condition (b) there is an integer i_2 ($i_2=1, 2, 3$), a component X_2 of $f^{-1}(|\sigma_{i_2}|)$, and a complementary domain o_2 of X_2 such that $o_2 \subset o_1$ and o_2 is essential mod $|\sigma_2|$. Continuing in this fashion we must arrive at the required pair (X, o) after a finite number of steps.

LEMMA 9. Let $\sigma_1, \sigma_2, \sigma_3$ be three N -simplexes of T whose geometric realizations are pairwise disjoint. Let p_1, p_2, p_3 be interior points of $|\sigma_1|, |\sigma_2|, |\sigma_3|$ respectively. Then for some permutation (i, j, k) of $(1, 2, 3)$ there is a component X of $f^{-1}(|\sigma_i|)$, a complementary domain o of X , and points p'_j, p'_k of o such that:

- (a) $f(p'_j) = p_j$,
- (b) $f(p'_k) = p_k$,
- (c) $f^{-1}(|\sigma_j|)$ does not separate p'_k from X ,
- (d) $f^{-1}(|\sigma_k|)$ does not separate p'_j from X .

Proof. Let i, o, X be the i_0, o, X of Lemma 8. Let j, k be any permutation of the remaining integers. Let $Q_1, Q_2, \dots, Q_w$ be those components of $f^{-1}(|\sigma_j|)$ in o which are not separated from X by any other component of $f^{-1}(|\sigma_j|)$. For $1 \leq t \leq w$ let W_t be the union of the complementary domains of Q_t not containing X . Then the W_t are pairwise disjoint. From condition (b) of Lemma 8, each component of W_t is inessential mod $|\sigma_j|$. Hence by Lemma 6, each W_t is inessential mod $|\sigma_j|$. Again, by Lemma 6, $\bigcup_1^w W_t$ is inessential mod $|\sigma_j|$. Hence there is a homotopy $F: S^N \times I \rightarrow S^N$ such that $F_0 = f, F_1| (S^N - \bigcup_1^w W_t) = f$, and $F_1(\bigcup_1^w W_t) \subset |\sigma_j|$. By Lemma 5, since o is essential mod $|\sigma_i|$, there is a point $p'_k \in o$ such that $F_1(p'_k) = p_k$. Since $F_1(\bigcup_1^w W_t) \subset |\sigma_j|$, $p'_k \notin \bigcup_1^w W_t$. Hence no Q_t (and therefore no component of $f^{-1}(|\sigma_j|)$) separates p'_k from X . It follows from the unicoherence⁽¹⁰⁾ of S^N that $f^{-1}(|\sigma_j|)$ does not separate p'_k from X . In the same manner we can find a $p'_j \in o \cap f^{-1}(p_j)$ such that $f^{-1}(|\sigma_k|)$ does not separate p'_j from X .

LEMMA 10. Let T be a triangulation of S^N ($N > 1$) and $\tilde{f}: T^n \rightarrow T^m$ a chain map such that the induced map $f: |T^n| \rightarrow |T^m|$ is essential. Suppose $\sigma_1, \sigma_2, \sigma_3$ are N -simplexes of T whose geometric realizations are pairwise disjoint. Let p_1, p_2, p_3 be points interior to $|\sigma_1|, |\sigma_2|, |\sigma_3|$ respectively. Finally, suppose $\epsilon < \min_i D(p_i, S^N - |\sigma_i|)$. Then f is not ϵ -crooked.

Proof. The hypotheses of this lemma include those of Lemma 9. Rather than restating the conclusions of Lemma 9 let us bodily incorporate its con-

⁽¹⁰⁾ It is here that the proof breaks down for $N=1$.

clusions and notation. It follows from conditions (c) and (d) and from the unicoherence of S^N that there exist arcs $[p'_j x_1]$ and $[p'_k x_2]$ in S^N such that $x_1 \cup x_2 \subset X$, $[p'_j x_1] \cap f^{-1}(|\sigma_k|) = [p'_k x_2] \cap f^{-1}(|\sigma_j|) = 0$, $[p'_k x_2] \cap X = x_2$, and $[p'_j x_1] \cap X = x_1$. Since X is a connected polyhedron there is an arc $[x_1 x_2]$ in X . Let h_1, h_2, h_3 be homeomorphisms of I into $[p'_j x_1]$, $[x_1 x_2]$, and $[x_2 p'_k]$ respectively, such that $h_1(0) = p'_j$, $h_2(0) = x_1$, $h_3(0) = x_2$. Let $g: I \rightarrow S^N$ be defined by:

$$g(t) = \begin{cases} h_1(3t), & 0 \leq t \leq 1/3, \\ h_2(3t - 1), & 1/3 \leq t \leq 2/3, \\ h_3(3t - 2), & 2/3 \leq t \leq 1. \end{cases}$$

If f were ϵ -crooked there would be real numbers t_1, t_2 such that $t_1 \leq t_2$, $|fg(0) - fg(t_2)| < \epsilon$, and $|fg(t_1) - fg(1)| < \epsilon$. Now $fg(0) = p_j$ and $fg(1) = p_k$.

Since $([p'_j x_1] \cup [x_1 x_2]) \cap f^{-1}(|\sigma_k|) = 0$, $|fg(t) - p_k| > \epsilon$ for $0 \leq t \leq 2/3$. Hence $t_1 > 2/3$. Similarly $|fg(t) - p_j| > \epsilon$ for $1/3 \leq t \leq 1$. Hence $t_2 < 1/3$. But this contradicts the assumption that $t_1 \leq t_2$. Hence f is not ϵ -crooked.

THEOREM 1. *Let $N > 1$. Then there is an $\epsilon > 0$ such that no essential map of S^N upon itself is ϵ -crooked.*

Proof. Let T be a triangulation of S^N fine enough to insure the existence of three N -simplexes $\sigma_1, \sigma_2, \sigma_3$ whose geometric realizations are pairwise disjoint. Let p_i be an interior point of $|\sigma_i|$ ($i = 1, 2, 3$) and let

$$2\epsilon < \min_i D(p_i, S^N - |\sigma_i|).$$

Suppose now that g is any essential map of S^N upon itself. By the Simplicial Approximation Theorem there are barycentric subdivisions T^n, T^m of T and a chain map $\bar{f}: T^n \rightarrow T^m$ such that if f is the map of $|T^n| \rightarrow |T^m|$ induced by $\bar{f}$, then f is homotopic to g and $\|f - g\| < \epsilon/2$. Suppose g is ϵ -crooked. Then by Lemma 4 f is 2ϵ -crooked. But by Lemma 10 f cannot be 2ϵ -crooked. Hence g cannot be ϵ -crooked.

THEOREM 2. *Let $X = \lim(X_i, f_i)$ where for some $N > 1$ each X_i is an N -sphere and f_i is essential. Then X is not hereditarily indecomposable.*

Proof. This theorem is a direct consequence of Theorem 1 and Lemma 3.

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